

# SOME THOUGHTS ABOUT $\mathbb{R}$

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ABSTRACT. We divide the set of real numbers  $\mathbb{R}$  in two subsets. A countable subset  $\mathbb{R}^-$  and an uncountable subset  $\mathbb{R}^+$ . We ask whether  $\mathbb{R}^-$  satisfies the definition of  $\mathbb{R}$ . This article is definitely not serious math. It's just a thought experiment and more for the fun than for the science.

## 1. $\mathbb{R}^-$ AND $\mathbb{R}^+$

**Definition 1.** We denote for any  $r \in \mathbb{R}$  with  $\alpha(r)$  the smallest number of characters needed to uniquely determine  $r$ .

In this text we're only interested in the existence of  $\alpha(r)$  for any given  $r \in \mathbb{R}$ , i.e. is  $\alpha(r) < \infty$ ?

**Example 1.**  $\pi$ ,  $e$  and  $\sqrt{2}$

- It is clear that  $\alpha(\pi) < \infty$  since  $\pi$  can be determined with the sum  $\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$  which can obviously be written with a finite number of characters.
- Along the same lines it is clear that  $\alpha(e) < \infty$  since  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ .
- $\alpha(\sqrt{2}) < \infty$  since  $\sqrt{2}$  can be uniquely determined as the positive solution of  $x^2 - 2 = 0$

All the numbers mentioned above can be uniquely determined with a finite number of characters. It doesn't matter whether special math characters like  $\sum$  count as one characters or have to be denoted with a term like "Sum".

With this definition we define two disjoint subsets of  $\mathbb{R}$ :

**Definition 2.** The set  $\mathbb{R}^-$  of sub-transcendental numbers is defined as

$$\mathbb{R}^- := \{r \in \mathbb{R} | \alpha(r) < \infty\}$$

**Definition 3.** The set  $\mathbb{R}^+$  of super-transcendental numbers is defined as

$$\mathbb{R}^+ := \mathbb{R} \setminus \mathbb{R}^-$$

## 2. PROPERTIES OF $\mathbb{R}^-$

**Proposition 1.**  $\mathbb{R}^-$  is countable.

*Proof.* This follows directly out of the fact that  $\mathbb{R}^-$  is a set of strings with a finite length. You can order them lexicographically and define an order over  $\mathbb{R}^-$ .  $\square$

It's quite simple to prove that  $\mathbb{R}^-$  is a strictly ordered set.  $\mathbb{R}^-$  inherits this property from  $\mathbb{R}$ .

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**Proposition 2.**  $(\mathbb{R}^-, +, \cdot)$  is a field.

*Proof.* We show only the existence of a multiplicative inverse for any given  $r \in \mathbb{R}^-$ . The remaining field properties can be shown in the same way:

For any number  $r \in \mathbb{R}^-$  there is a uniquely determined number  $r^{-1} \in \mathbb{R}$  such that  $rr^{-1} = 1$ . Since  $r^{-1}$  is uniquely defined it is:  $r^{-1} \in \mathbb{R}^-$ . In the same way all field properties of  $\mathbb{R}^-$  can be shown.  $\square$

**Proposition 3.**  $\leq$  is a linear order over  $\mathbb{R}^-$ .  $\leq$  is compatible with the addition and the multiplication.

*Proof.* This is a direct conclusion from  $\mathbb{R}^- \subset \mathbb{R}$ .  $\square$

### 3. COMING UP NEXT

We'll show that  $\mathbb{R}^-$  fulfills the completeness axiom, i.e. any give subset of  $\mathbb{R}^-$  has an infimum in  $\mathbb{R}^-$  (which concludes from the fact that the infimum  $i$  of a given set  $M \subset \mathbb{R}^-$  is uniquely determined, hence  $i \in \mathbb{R}^-$ ). With propositions 2 and 3  $\mathbb{R}^-$  fulfills all the axioms for the real numbers. Hence

$$(1) \quad \mathbb{R}^- = \mathbb{R}$$